

FISHER'S FUNDAMENTAL THEOREM OF NATURAL SELECTION¹

Genotype	A_1A_1	A_1A_2	A_2A_2
Frequency	p^2	$2pq$	q^2
Fitness	w_{11}	w_{12}	w_{22}
Additive fitness value	$2\alpha_1$	$\alpha_1 + \alpha_2$	$2\alpha_2$

A well-known fact about viability selection at one locus:

$$\Delta p = \left(\frac{pq}{2\bar{w}} \right) \left(\frac{d\bar{w}}{dp} \right) .$$

Another well-known fact (from calculus, Taylor's Theorem):

$$f(x) = f(a) + \sum_{k=1}^{\infty} \left(\frac{(x-a)^k}{k!} \right) f^{(k)}(a) .$$

It follows immediately that

$$\bar{w}' = \bar{w} + (\Delta p) \left(\frac{d\bar{w}}{dp} \right) + \left(\frac{(\Delta p)^2}{2} \right) \left(\frac{d^2\bar{w}}{dp^2} \right) .$$

Or, equivalently

$$\Delta\bar{w} = (\Delta p) \left(\frac{d\bar{w}}{dp} \right) + \left(\frac{(\Delta p)^2}{2} \right) \left(\frac{d^2\bar{w}}{dp^2} \right) . \tag{1}$$

Recalling that $\bar{w} = p^2w_{11} + 2p(1-p)w_{12} + (1-p)^2w_{22}$ we find that

$$\begin{aligned} \frac{d\bar{w}}{dp} &= 2pw_{11} + 2(1-p)w_{12} - 2pw_{12} - 2(1-p)w_{22} \\ &= 2[(pw_{11} + qw_{12}) - (pw_{12} + qw_{22})] \\ &= 2[(pw_{11} + qw_{12} - \bar{w}/2) - (pw_{12} + qw_{22} - \bar{w}/2)] \\ &= 2[\alpha_1 - \alpha_2] \\ &= 2\alpha \quad , \end{aligned}$$

where the last two steps use the definitions for α_1 and α_2 , and we set $\alpha = \alpha_1 - \alpha_2$. Similarly,

$$\begin{aligned} \frac{d^2\bar{w}}{dp^2} &= 2w_{11} - 2w_{12} - 2w_{12} + 2w_{22} \\ &= 2(w_{11} - 2w_{12} + w_{22}) \end{aligned}$$

Now we are ready to make some substitutions in (1), namely

¹ proof due to C. C. Li

$$\begin{aligned}
\Delta\bar{w} &= \left\{ \left(\frac{pq}{2\bar{w}} \right) \left(\frac{d\bar{w}}{dp} \right) \right\} \left(\frac{d\bar{w}}{dp} \right) + \frac{\left\{ \left(\frac{pq}{2\bar{w}} \right) \left(\frac{d\bar{w}}{dp} \right) \right\}^2}{2} [2(w_{11} - 2w_{12} + w_{22})] \\
&= \left\{ \left(\frac{pq}{2\bar{w}} \right) (2\alpha) \right\} (2\alpha) + \left\{ \left(\frac{pq}{2\bar{w}} \right) (2\alpha) \right\}^2 (w_{11} - 2w_{12} + w_{22}) \\
&= \frac{2pq\alpha^2}{\bar{w}} + \frac{p^2q^2\alpha^2}{\bar{w}^2} (w_{11} - 2w_{12} + w_{22}) \\
&= \frac{V_a}{\bar{w}} \left\{ 1 + \frac{pq}{2\bar{w}} (w_{11} - 2w_{12} + w_{22}) \right\} \quad ,
\end{aligned} \tag{2}$$

where the last step follows from the observation that $V_a = 2pq\alpha^2$. The quantity $\frac{pq}{2\bar{w}}(w_{11} - 2w_{12} + w_{22})$ is usually quite small, especially if selection is not too intense. Thus, (2) can be further simplified to

$$\Delta\bar{w} \approx \frac{V_a}{\bar{w}} \quad \text{Q.E.D.}$$

PHENOTYPIC RESPONSE TO SELECTION

Remember Fisher's Fundamental Theorem? Well let's consider the evolution of the mean phenotype in a population. Before we can do this, however, we need to think a bit more carefully about the relationship between genotype, phenotype, and fitness. Let $F_{ij}(x)$ be the probability that genotype A_iA_j has a phenotype smaller than x . Then x_{ij} , the genotypic value of A_iA_j is

$$x_{ij} = \int_{-\infty}^{\infty} x dF_{ij}(x)$$

and the population mean phenotype is $p^2x_{11} + 2pqx_{12} + q^2x_{22}$. If an individual with phenotype x has fitness $w(x)$, then

$$w_{ij} = \int_{-\infty}^{\infty} w(x) dF_{ij}(x)$$

and $\bar{w} = p^2w_{11} + 2pqw_{12} + q^2w_{22}$. Using our old friend, the Taylor series expansion,²

$$\begin{aligned} \bar{x}' &= \bar{x} + (\Delta p) \left(\frac{d\bar{x}}{dp} \right) + O(p^2) \\ \bar{x} &= p^2x_{11} + 2pqx_{12} + q^2x_{22} \\ \frac{d\bar{x}}{dp} &= 2px_{11} + 2qx_{12} - 2px_{12} - 2qx_{22} \\ &= 2 \{ (px_{11} + qx_{12} - \bar{x}/2) + (px_{12} + qx_{22} - \bar{x}/2) \} \\ &= 2(\alpha_1 - \alpha_2) \\ \bar{x}' &\approx \bar{x} + (\Delta p) (2(\alpha_1 - \alpha_2)) \\ \Delta\bar{x} &= (\Delta p) (2(\alpha_1 - \alpha_2)) \end{aligned} \tag{3}$$

Remember (from lo those many weeks ago) that

$$p' = \frac{p^2w_{11} + pqw_{12}}{\bar{w}} .$$

Thus,

$$\begin{aligned} \Delta p &= p' - p \\ &= \frac{p^2w_{11} + pqw_{12}}{\bar{w}} - p \\ &= \frac{p^2w_{11} + pqw_{12} - p\bar{w}}{\bar{w}} \\ &= p \left(\frac{pw_{11} + qw_{12} - \bar{w}}{\bar{w}} \right) \end{aligned} \tag{4}$$

Now,³ let's do a linear regression of fitness on phenotype. After all, to make any further progress, we need to relate phenotype to fitness, so that we can use the relationship between

² Isn't it nice to encounter old friends in new surroundings?

³ Since we're having so much fun with mathematics why should we stop here?.

phenotype and genotype to infer the change in allele frequencies, from which we will infer the change in mean phenotype.⁴ From statistics, we know that the slope of this regression line is

$$\beta_1 = \frac{Cov(w, x)}{Var(x)} \quad (5)$$

and its intercept is

$$\beta_0 = \bar{w} - \beta_1 \bar{x} \quad .$$

Let's use this regression equation to determine the fitness of each genotype. This is only an approximation to the true fitness,⁵ but it is adequate for many purposes.

$$\begin{aligned} w_{ij} &= \int_{-\infty}^{\infty} w(x) dF_{ij}(x) \\ &\approx \int_{-\infty}^{\infty} (\beta_0 + \beta_1 x) dF_{ij}(x) \\ &= \beta_0 + \beta_1 x_{ij} \\ \bar{w} &= \beta_0 + \beta_1 \bar{x} \end{aligned}$$

Thus, from (4)

$$\begin{aligned} \Delta p &= p \left(\frac{pw_{11} + qw_{12} - \bar{w}}{\bar{w}} \right) \\ &= p \left(\frac{p(\beta_0 + \beta_1 x_{11}) + q(\beta_0 + \beta_1 x_{12}) - (\beta_0 + \beta_1 \bar{x})}{\bar{w}} \right) \\ &= p\beta_1 \left(\frac{px_{11} + qx_{12} - \bar{x}}{\bar{w}} \right) \\ &= p\beta_1 \left(\frac{\alpha_1 - \bar{x}/2}{\bar{w}} \right) \\ &= p\beta_1 \left(\frac{\alpha_1 - (p\alpha_1 + q\alpha_2)}{\bar{w}} \right) \\ &= \frac{pq\beta_1(\alpha_1 - \alpha_2)}{\bar{w}} \end{aligned}$$

So where are we now?⁶ From (3)

$$\begin{aligned} \Delta \bar{x} &= (\Delta p) (2(\alpha_1 - \alpha_2)) \\ &= \left(\frac{pq\beta_1(\alpha_1 - \alpha_2)}{\bar{w}} \right) (2(\alpha_1 - \alpha_2)) \\ &= 2pq\alpha^2 \left(\frac{\beta_1}{\bar{w}} \right) \\ &= V_a \left(\frac{\beta_1}{\bar{w}} \right) \end{aligned} \quad (6)$$

⁴ Whew! That was a mouthful.

⁵ We are assuming (implicitly) that the fitnesses are adequately approximated by a linear function of our phenotypic measure.

⁶ You don't have to tell me where you *wish* you were. I can reliably guess that it's not here.

This is great if we've done the regression between fitness and phenotype, but what if we haven't?⁷ Let's look at $Cov(w, x)$ in a little more detail.

$$\begin{aligned}
Cov(w, x) &= p^2 \int_{-\infty}^{\infty} xw(x)dF_{11}(x) + 2pq \int_{-\infty}^{\infty} xw(x)dF_{12}(x) \\
&\quad + q^2 \int_{-\infty}^{\infty} xw(x)dF_{22}(x) - \bar{x}\bar{w} \\
&= p^2(Cov_{11}(w, x) + x_{11}\bar{w}) + 2pq(Cov_{12}(w, x) + x_{12}\bar{w}) \\
&\quad + q^2(Cov_{22}(w, x) + x_{22}\bar{w}) - \bar{x}\bar{w} \\
&= p^2Cov_{11}(w, x) + 2pqCov_{12}(w, x) + q^2Cov_{22}(w, x)
\end{aligned} \tag{7}$$

$$\begin{aligned}
Cov_{ij}(w, x) &= \int_{-\infty}^{\infty} xw(x)dF_{ij}(x) - x_{ij}\bar{w} \\
&= \bar{w} \left(\int_{-\infty}^{\infty} \frac{xw(x)}{\bar{w}} dF_{ij}(x) - x_{ij} \right) \quad , \\
&= \bar{w}(x_{ij}^* - x_{ij})
\end{aligned} \tag{8}$$

where x_{ij}^* refers to the mean phenotype of A_iA_j after selection. Combining (7) and (8),

$$Cov(w, x) = \bar{w}(\bar{x}^* - \bar{x}) \quad , \tag{9}$$

where \bar{x}^* is the population mean phenotype after selection. In short,⁸ combining (5), (6), and (9)

$$\begin{aligned}
\Delta\bar{x} &= V_a \left(\frac{\bar{w}(\bar{x}^* - \bar{x})}{\bar{w}} \right) \\
&= h_N^2(\bar{x}^* - \bar{x})
\end{aligned} \tag{10}$$

$\Delta\bar{x}$ is referred to as the response to selection and is often given the symbol R . It is the change in population mean between parents (before selection) and their offspring (before selection). $\bar{x}^* - \bar{x}$ is referred to as the selection differential and is often given the symbol S . It is the difference between the mean phenotype in reproductive individuals and the mean phenotype in the population as a whole. Thus, (10) can be summarized as

$$R = h_N^2 S \quad .$$

⁷ Just hang on a little while longer. We're almost there.

⁸ We finally made it.

A NUMERICAL EXAMPLE

Genotype	A_1A_1	A_1A_2	A_2A_2
Phenotype	1.303	1.249	0.948

Given these phenotypes, $p = 0.25$, and $V_p = 0.16$, it follows that $\bar{x} = 1.08$ and $h_N^2 = 0.1342$. Suppose the mean phenotype after selection is 1.544. What will the phenotype be among the newly born progeny?

$$\begin{aligned} S &= \bar{x}^* - \bar{x} \\ &= 1.544 - 1.08 \\ &= 0.464 \\ \Delta\bar{x} &= h_N^2 S \\ &= (0.1342)(0.464) \\ &= 0.06 \\ \bar{x}' &= \bar{x} + \Delta\bar{x} \\ &= 1.08 + 0.06 \\ &= 1.14 \end{aligned}$$