

ANALYZING THE GENETIC STRUCTURE OF POPULATIONS

Introduction

So far we've focused on inbreeding as one important way that populations may fail to mate at random, but there's another way in which virtually all populations and species fail to mate at random. Individuals tend to mate with those that are nearby. Even within a fairly small area, phenomena like nearest neighbor pollination in flowering plants or home-site fidelity in animals can cause mates to be selected in a geographically non-random way. What are the population genetic consequences of this form of non-random mating?

Well, if you think about it a little, you can probably figure it out. Since individuals that occur close to one another tend to be more genetically similar than those that occur far apart, the impacts of local mating will mimic those of inbreeding within a single, well-mixed population.

A numerical example

For example, suppose we have two subpopulations of green lacewings, one of which occurs in forests the other of which occurs in adjacent meadows.¹ Suppose further that within each subpopulation mating occurs completely at random, but that there is no mating between forest and meadow individuals. Suppose we've determined allele frequencies in each population at a locus coding for phosphoglucosomerase (*PGI*), which conveniently has only two alleles. The frequency of A_1 in the forest is 0.4 and in the meadow in 0.7. We can easily calculate the expected genotype frequencies within each population, namely

	A_1A_1	A_1A_2	A_2A_2
Forest	0.16	0.48	0.36
Meadow	0.49	0.42	0.09

¹Those of you who've been in EEB for a while will know that these are probably different species, but humor me, and forget that you know that.

Suppose, however, we were to consider a combined population consisting of 100 individuals from the forest subpopulation and 100 individuals from the meadow subpopulation. Then we'd get the following:²

	A_1A_1	A_1A_2	A_2A_2
From forest	16	48	36
From meadow	49	42	9
Total	65	90	45

So the frequency of A_1 is $(2(65) + 90)/(2(65 + 90 + 45)) = 0.55$. Notice that this is just the average allele frequency in the two subpopulations, i.e., $(0.4 + 0.7)/2$. Since each subpopulation has genotypes in Hardy-Weinberg proportions, you might expect the combined population to have genotypes in Hardy-Weinberg proportions, but if you did you'd be wrong. Just look.

	A_1A_1	A_1A_2	A_2A_2
Expected (from $p = 0.55$)	$(0.3025)200$ 60.5	$(0.4950)200$ 99.0	$(0.2025)200$ 40.5
Observed (from table above)	65	90	45

The expected and observed don't match, even though there is random mating within both subpopulations. They don't match because there isn't random mating *in the combined population*, only within each subpopulation. Forest lacewings choose mates at random from other forest lacewings, but they never mate with a meadow lacewing (and *vice versa*). Our sample includes two populations that don't mix. As a result, heterozygotes in our combined sample are less frequent (0.45 vs 0.495) than we'd expect if the population were well mixed with an allele frequency of 0.55. This is an example of what's known as the *Wahlund effect* [8].

The algebraic development

Even though you've only known me for a couple of weeks now, you should know me well enough to know that I'm not going to be satisfied with a numerical example. You should know that I now feel the need to do some algebra to describe this situation a little more generally.

²If we ignore sampling error.

Suppose we know allele frequencies in k subpopulations.³ Let p_i be the frequency of A_1 in the i th subpopulation. Then if we assume that all subpopulations contribute equally to combined population,⁴ we can calculate expected and observed genotype frequencies the way we did above:

	A_1A_1	A_1A_2	A_2A_2
Expected	\bar{p}^2	$2\bar{p}\bar{q}$	\bar{q}^2
Observed	$\frac{1}{k} \sum p_i^2$	$\frac{1}{k} \sum 2p_iq_i$	$\frac{1}{k} \sum q_i^2$

where $\bar{p} = \sum p_i/k$ and $\bar{q} = 1 - \bar{p}$ are the average allele frequencies in the combined sample. Now

$$\frac{1}{k} \sum p_i^2 = \frac{1}{k} \sum (p_i - \bar{p} + \bar{p})^2 \quad (1)$$

$$= \frac{1}{k} \sum \left((p_i - \bar{p})^2 + 2\bar{p}(p_i - \bar{p}) + \bar{p}^2 \right) \quad (2)$$

$$= \frac{1}{k} \sum (p_i - \bar{p})^2 + \bar{p}^2 \quad (3)$$

$$= \text{Var}(p) + \bar{p}^2 \quad (4)$$

Similarly,

$$\frac{1}{k} \sum 2p_iq_i = 2\bar{p}\bar{q} - 2\text{Var}(p) \quad (5)$$

$$\frac{1}{k} \sum q_i^2 = \bar{q}^2 + \text{Var}(p) \quad (6)$$

Since $\text{Var}(p) \geq 0$ by definition, with equality holding only when all subpopulations have the same allele frequency, we can conclude that

- Homozygotes will be more frequent and heterozygotes will be less frequent than expected based on the allele frequency in the combined population.
- The magnitude of the departure from expectations is directly related to the magnitude of the variance in allele frequencies across populations, $\text{Var}(p)$.

³For the time being, I'm going to assume that we know the allele frequencies without error, i.e., that we didn't have to estimate them from data. We'll deal with real life, i.e., how we can detect the Wahlund effect when we have to *estimate* allele frequencies from data, a little later.

⁴We'd get the same result by relaxing this assumption, but the algebra gets messier, so why bother?

- The effect will apply to *any* mixing of samples in which the subpopulations combined have different allele frequencies.⁵
- The same general phenomenon will occur if there are multiple alleles at a locus, although it is possible for one or a few heterozygotes to be *more* frequent than expected if there is positive covariance in the constituent allele frequencies across populations.⁶
- The effect is analogous to inbreeding. Homozygotes are more frequent and heterozygotes are less frequent than expected.⁷

To return to our earlier numerical example:

$$\begin{aligned} \text{Var}(p) &= \left((0.4 - 0.55)^2 + (0.7 - 0.55)^2 \right) / 2 & (7) \\ &= 0.0225 & (8) \end{aligned}$$

	Expected			Observed	
A_1A_1	0.3025	+	0.0225	=	0.3250
A_1A_2	0.4950	-	$2(0.0225)$	=	0.4500
A_2A_2	0.2025	+	0.0225	=	0.2250

Wright's F -statistics

One limitation of the way I've described things so far is that $\text{Var}(p)$ doesn't provide a convenient way to compare population structure from different samples. $\text{Var}(p)$ can be much larger if both alleles are about equally common in the whole sample than if one occurs at a mean frequency of 0.99 and the other at a frequency of 0.01. Moreover, if you stare at equations (4)–(6) for a while, you begin to realize that they look a lot like some equations we've already encountered. Namely, if we were to define F_{st} ⁸ as $\text{Var}(p)/\bar{p}\bar{q}$, then we could

⁵For example, if we combine samples from different years or across age classes of long-lived organisms, we may see a deficiency of heterozygotes in the sample purely as a result of allele frequency differences across years. Remember that I told you one of the assumptions underlying derivation of the Hardy-Weinberg principle is that generations are non-overlapping? This is why.

⁶If you're curious about this, feel free to ask, but I'll have to dig out my copy of Li [4] to answer. I don't carry those details around in my head.

⁷And this is what we predicted when we started.

⁸The reason for the subscript will become apparent later. It's also *very* important to notice that I'm defining F_{ST} here in terms of the population parameters p and $\text{Var}(p)$. Again, we'll return to the problem of how to *estimate* F_{ST} from data a little later.

rewrite equations (4)–(6) as

$$\frac{1}{k} \sum p_i^2 = \bar{p}^2 + F_{st} \bar{p} \bar{q} \quad (9)$$

$$\frac{1}{k} \sum 2p_i q_i = 2\bar{p} \bar{q} (1 - F_{st}) \quad (10)$$

$$\frac{1}{k} \sum q_i^2 = \bar{q}^2 + F_{st} \bar{p} \bar{q} \quad (11)$$

And it’s not even completely artificial to define F_{st} the way I did. After all, the effect of geographic structure is to cause matings to occur among genetically similar individuals. It’s rather like inbreeding.⁹ Moreover, the extent to which this local mating matters depends on the extent to which populations differ from one another. It turns out that $\bar{p} \bar{q}$ is the maximum allele frequency variance possible, given the observed mean frequency. So one way of thinking about F_{st} is that it measures the amount of allele frequency variance in a sample relative to the maximum possible.¹⁰

There may, of course, be inbreeding within populations, too. But it’s easy to incorporate this into the framework, too.¹¹ Let H_i be the actual heterozygosity in individuals within subpopulations, H_s be the expected heterozygosity within subpopulations assuming Hardy-Weinberg within populations, and H_t be the expected heterozygosity in the combined population assuming Hardy-Weinberg over the whole sample.¹² Then thinking of f as a measure of departure from Hardy-Weinberg and assuming that all populations depart from Hardy-Weinberg to the same degree, i.e., that they all have the same f , we can define

$$F_{it} = 1 - \frac{H_i}{H_t} .$$

F_{it} is the overall departure from Hardy-Weinberg in the entire sample. Let’s fiddle with F_{ST}

⁹To be precise, it is a form of positive assortative mating in which the choice of mates is based on geographical proximity.

¹⁰I say “one way”, because there are several other ways to talk about F_{st} , too. But we won’t talk about them until later.

¹¹At least it’s easy once you’ve been shown how.

¹²Please remember that we’re assuming we know those frequencies exactly. In real applications, of course, we’ll *estimate* those frequencies from data, so we’ll have to account for sampling error when we actually try to estimate these things. If you’re getting the impression that I think the distinction between allele frequencies as *parameters*, i.e., the real allele frequency in the population, and allele frequencies as *estimates*, i.e., the sample frequencies from which we hope to estimate the parameters, is really important, you’re getting the right impression.

a bit.¹³

$$\begin{aligned} 1 - F_{it} &= \frac{H_i}{H_t} \\ &= \left(\frac{H_i}{H_s}\right) \left(\frac{H_s}{H_t}\right) \\ &= (1 - F_{is})(1 - F_{st}) \quad , \end{aligned}$$

where F_{is} is the inbreeding coefficient within populations, i.e., f , and F_{st} has the same definition as before.¹⁴ H_t is often referred to as the genetic diversity in a population. So another way of thinking about $F_{st} = (H_t - H_s)/H_t$ is that it's the proportion of the diversity in the sample that's due to allele frequency differences among populations.

Estimating F -statistics

We've now seen the principles underlying Wright's F -statistics. I should point out that Gustave Malécot developed very similar ideas at about the same time as Wright, but since Wright's notation stuck,¹⁵ population geneticists generally refer to statistics like those we've discussed as Wright's F -statistics.¹⁶

Neither Wright nor Malécot worried too much about the problem of estimating F -statistics from data. Both realized that any inferences about population structure are based on a sample and that the characteristics of the sample may differ from those of the population from which it was drawn, but neither developed any explicit way of dealing with those differences. Wright develops some very *ad hoc* approaches in his book [12], but they have been forgotten, which is good because they aren't satisfactory and they shouldn't be used. There are now three reasonable approaches available:¹⁷

1. Nei's G -statistics,

¹³Are you beginning to see how peculiar I am? Do you know anyone else who gets a kick out of playing around with formulas and equations.

¹⁴It takes a fair amount of algebra to show that this definition of F_{st} is equivalent to the one I showed you before, so you'll just have to take my word for it.

¹⁵Probably because he published in English and Malécot published in French.

¹⁶The Hardy-Weinberg proportions should probably be referred to as the Hardy-Weinberg-Castle proportions too, since Castle pointed out the same principle. For some reason, though, his demonstration didn't have the impact that Hardy's and Weinberg's did. So we generally talk about the Hardy-Weinberg principle.

¹⁷And as we'll soon see, I'm not too crazy about one of these three. To my mind, there are really only two approaches that anyone should consider, and those two approaches are really just variants of the same basic idea.

2. Weir and Cockerham's θ -statistics, and
3. A Bayesian analog of θ .¹⁸

An example from *Isotoma petraea*

To make the differences in implementation and calculation clear, I'm going to use data from 12 populations of *Isotoma petraea* in southwestern Australia surveyed for genotype at *GOT-1* [3] as an example throughout these discussions (Table 1).

Let's ignore the sampling problem for a moment and calculate the F -statistics as if we had observed the population allele frequencies without error. They'll serve as our baseline for comparison.

$$\begin{aligned}
 \bar{p} &= 0.8888 \\
 \text{Var}(p) &= 0.02118 \\
 F_{st} &= 0.2143 \\
 \text{Individual heterozygosity} &= (0.0000 + 0.1500 + 0.1000 + 0.0000 + 0.0000 + 0.1667 + 0.1000 \\
 &\quad + 0.0909 + 0.0000 + 0.0000 + 1.0000 + 0.0000)/12 \\
 &= 0.1340 \\
 \text{Expected heterozygosity} &= 2(0.8888)(1 - 0.8888) \\
 &= 0.1976 \\
 F_{it} &= 1 - \frac{\text{Individual heterozygosity}}{\text{Expected heterozygosity}} \\
 &= 1 - \frac{0.1340}{0.1976}
 \end{aligned}$$

¹⁸This is, as you have probably already guessed, my personal favorite. We don't have time to discuss it in lecture, but if you're interested, ask me about it. I should also tell you that Gregory Owens pointed out on Twitter that arguments about genetic differentiation can get a little heated (https://twitter.com/Greg_Owens/status/1582104811629346817). They may not turn into an actual fistfight, but there have been some pretty extreme statements made.



Gregory Owens, PhD
@Greg_Owens

...

Did you guys see that argument about genetic differentiation? It was bad, a Fst fight almost broke out.

4:22 PM · Oct 17, 2022 · Twitter Web App

Population	Genotype			\hat{p}
	A_1A_1	A_1A_2	A_2A_2	
Yackeyackine Soak	29	0	0	1.0000
Gnarlbine Rock	14	3	3	0.7750
Boorabbin	15	2	3	0.8000
Bullabulling	9	0	0	1.0000
Mt. Caudan	9	0	0	1.0000
Victoria Rock	23	5	2	0.8500
Yellowdine	23	3	4	0.8167
Wargangering	29	3	1	0.9242
Wagga Rock	5	0	0	1.0000
“Iron Knob Major”	1	0	0	1.0000
Rainy Rocks	0	1	0	0.5000
“Rainy Rocks Major”	1	0	0	1.0000

Table 1: Genotype counts at the *GOT* – 1 locus in *Isotoma petraea* (from [3]).

$$\begin{aligned}
&= 0.3221 \\
1 - F_{it} &= (1 - F_{is})(1 - F_{st}) \\
F_{is} &= \frac{F_{it} - F_{st}}{1 - F_{st}} \\
&= \frac{0.3221 - 0.2143}{1 - 0.2143} \\
&= 0.1372
\end{aligned}$$

Summary

Correlation of gametes due to inbreeding within subpopulations (F_{is}):	0.1372
Correlation of gametes within subpopulations (F_{st}):	0.2143
Correlation of gametes in sample (F_{it}):	0.3221

Why do I refer to them as the “correlation of gametes ...”? There are two reasons:

1. That’s the way Wright always referred to and interpreted them.
2. We can define indicator variables $x_{ijk} = 1$ if the i th allele in the j th individual of population k is A_1 and $x_{ijk} = 0$ if that allele is not A_1 . This may seem like a strange

thing to do, but the Weir and Cockerham approach to F -statistics described below uses just such an approach. If we do this, then the definitions for F_{is} , F_{st} , and F_{it} follow directly.¹⁹

Notice that F_{is} could be negative, i.e., there could be an *excess* of heterozygotes within populations ($F_{is} < 0$). Notice also that we're implicitly assuming that the extent of departure from Hardy-Weinberg proportions is the same in all populations. Equivalently, we can regard F_{is} as the *average* departure from Hardy-Weinberg proportions across all populations.

Statistical expectation and unbiased estimates

So far I've assumed that we know the allele frequencies without error, but of course that's never the case unless we've created experimental populations. We are always taking a sample from a population and inferring — estimating — allele frequencies from our sample. Similarly, we are *estimating* F_{ST} and our *estimate* of F_{ST} needs to take account of the imprecision in the allele frequency estimates on which it was based. To understand one approach to dealing with this uncertainty I need to introduce two new concepts: *statistical expectation* and *unbiased estimates*.

The concept of statistical expectation is actually quite an easy one. It is an arithmetic average, just one calculated from probabilities instead of being calculated from samples. So, for example, let $P(k|p, N)$ be the probability that we find k A_1 alleles in our sample of size N given that the allele frequency in the population is p . Then the *expected number* of A_1 alleles in our sample is just

$$\begin{aligned} E(k) &= \sum_{k=0}^n kP(k|p, N) \\ &= np \end{aligned}$$

where n is the total number of alleles in our sample.²⁰

Now consider the expected value of our sample estimate of the population allele frequency, $\hat{p} = k/n$, where k now refers to the number of A_1 alleles we actually found.

$$E(\hat{p}) = E\left(\sum_{k=1}^n (k/n)\right)$$

¹⁹See [9] for details.

²⁰ $P(k|p, N) = \binom{N}{k} p^k (1-p)^{N-k}$. The algebra in getting from the first line to the second is a little complicated, but feel free to ask me about it if you're interested.

$$\begin{aligned}
&= \sum_{k=1}^n (k/n)P(k|p, N) \\
&= (1/n) \left(\sum_{k=1}^n kP(k|p, N) \right) \\
&= (1/n)(np) \\
&= p \quad .
\end{aligned}$$

Because $E(\hat{p}) = p$, \hat{p} is said to be an *unbiased estimate* of p .²¹ When an estimate is unbiased it means that if we were to repeat the sampling experiment an infinite number of times and to take the average of the estimates, the average of those values would be equal to the (unknown) parameter value.

What about estimating the frequency of heterozygotes within a population? The obvious estimator is $\tilde{H} = 2\hat{p}(1 - \hat{p})$. Well,

$$\begin{aligned}
E(\tilde{H}) &= E(2\hat{p}(1 - \hat{p})) \\
&= 2(E(\hat{p}) - E(\hat{p}^2)) \\
&= \text{TAMO} \\
&= ((n - 1)/n)2p(1 - p) \quad .
\end{aligned}$$

Because $E(\tilde{H}) \neq 2p(1 - p)$, \tilde{H} is a *biased estimate* of $2p(1 - p)$. If, however, we set $\hat{H} = (n/(n - 1))\tilde{H}$, however, \hat{H} is an unbiased estimator of $2p(1 - p)$.²²

If you've ever wondered why you typically divide the sum of squared deviations about the mean by $n - 1$ instead of n when estimating the variance of a sample, this is why. Dividing by n gives you a (slightly) biased estimator.

²¹Notice that I'm using a hat here to refer to a statistical estimate. Remember when I told you I'd be using hats for a couple of different purposes? Well, this is the second one.

²²If you're wondering how I got from the second equation for \hat{H} to the last one, ask me about it or read the gory details section that follows. TAMO is short for "Then a miracle occurs." You'll see that acronym repeatedly this semester.

The gory details²³

Starting where we left off above:

$$\begin{aligned} E(\tilde{H}) &= 2 \left(E(\hat{p}) - E(\hat{p}^2) \right) \\ &= 2 \left(p - E \left((k/n)^2 \right) \right) \quad , \end{aligned}$$

where k is the number of A_1 alleles in our sample and n is the sample size.

$$\begin{aligned} E \left((k/n)^2 \right) &= \sum (k/n)^2 P(k|p, N) \\ &= (1/n)^2 \sum k^2 P(k|p, N) \\ &= (1/n)^2 \left(\text{Var}(k) + \bar{k}^2 \right) \\ &= (1/n)^2 \left(np(1-p) + n^2 p^2 \right) \\ &= p(1-p)/n + p^2 \quad . \end{aligned}$$

Substituting this back into the equation above yields the following:

$$\begin{aligned} E(\tilde{H}) &= 2 \left(p - \left(p(1-p)/n + p^2 \right) \right) \\ &= 2 \left(p(1-p) - p(1-p)/n \right) \\ &= (1 - 1/n) 2p(1-p) \\ &= ((n-1)/n) 2p(1-p) \quad . \end{aligned}$$

Corrections for sampling error

There are two sources of allele frequency difference among subpopulations in our sample: (1) real differences in the allele frequencies among our sampled subpopulations and (2) differences that arise because allele frequencies in our samples differ from those in the subpopulations from which they were taken.²⁴

²³Skip this part unless you are *really, really* interested in how I got from the second equation to the third equation in the last paragraph. This is more likely to confuse you than help unless you know that the variance of a binomial sample is $np(1-p)$ and that $E(k^2) = \text{Var}(p) + p^2$.

²⁴There's actually a third source of error that we'll get to in a moment. The populations we're sampling from are the product of an evolutionary process, and since the populations aren't of infinite size, drift has played a role in determining allele frequencies in them. As a result, if we were to go back in time and re-run the evolutionary process, we'd end up with a different set of real allele frequency differences. We'll talk about this more in just a moment when we get to Weir and Cockerham's statistics.

Nei's G_{st}

Nei and Chesser [5] described one approach to accounting for sampling error. So far as I've been able to determine, there aren't any currently supported programs²⁵ that calculate the bias-corrected versions of G_{st} .²⁶ I calculated the results in Table 2 by hand.

The calculations are tedious, which is why you'll want to find some way of automating the calculations if you want to do them.²⁷

$$\begin{aligned} H_i &= 1 - \frac{1}{N} \sum_{k=1}^N \sum_{i=1}^m X_{kii} \\ H_s &= \frac{\tilde{n}}{\tilde{n} - 1} \left[1 - \sum_{i=1}^m \hat{x}_i^2 - \frac{H_I}{2\tilde{n}} \right] \\ H_t &= 1 - \sum_{i=1}^m \bar{x}_i^2 + \frac{H_S}{\tilde{n}} - \frac{H_I}{2\tilde{n}N} \end{aligned}$$

where we have N subpopulations, $\hat{x}_i^2 = \sum_{k=1}^N x_{ki}^2/N$, $\bar{x}_i = \sum_{k=1}^N x_{ki}/N$, \tilde{n} is the harmonic mean of the population sample sizes, i.e., $\tilde{n} = \frac{1}{\frac{1}{N} \sum_{k=1}^N \frac{1}{n_k}}$, X_{kii} is the frequency of genotype $A_i A_i$ in population k , x_{ki} is the frequency of allele A_i in population k , and n_k is the sample size from population k . Recall that

$$\begin{aligned} F_{is} &= 1 - \frac{H_i}{H_s} \\ F_{st} &= 1 - \frac{H_s}{H_t} \\ F_{it} &= 1 - \frac{H_i}{H_t} . \end{aligned}$$

Weir and Cockerham's θ

Weir and Cockerham [10] describe the fundamental ideas behind this approach. Weir and Hill [11] bring things up to date. Holsinger and Weir [2] provide a less technical overview.²⁸

²⁵Popgene estimates G_{st} , but I don't think it's been updated since 2000. FSTAT also estimates gene diversities, but the most recent version is from 2002.

²⁶There's a reason for this that we'll get to in a moment. It's alluded to in the footnote before the last one.

²⁷It is also one big reason why most people use Weir and Cockerham's θ . There's readily available software that calculates it for you.

²⁸We also talk a bit more about how F -statistics can be used. If you just can't get enough of this, I suggest you take a look at Verity and Nichols [7]. They provide a really solid analysis of F_{ST} , G_{ST} , and some related statistics.

Most, if not all, packages available now that estimate F_{ST} provide estimates of θ . The most important difference between θ and G_{st} and the reason why G_{st} has fallen into disuse is that G_{st} ignores an important source of sampling error that θ incorporates.

In many applications, especially in evolutionary biology, the subpopulations included in our sample are not an exhasutive sample of all populations. Moreover, even if we have sampled from every population there is now, we may not have sampled from every population there ever was. And even if we've sampled from every population there ever was, we know that there are random elements in any evolutionary process. Thus, if we could run the clock back and start it over again, the genetic composition of the populations we have might be rather different from that of the populations we sampled. In other words, our populations are, in many cases, best regarded as a random sample from a much larger set of populations that could have been sampled.

Even more gory details²⁹

Let $x_{mn,i}$ be an indicator variable such that $x_{mn,i} = 1$ if allele m from individual n is of type i and is 0 otherwise. Clearly, the sample frequency $\hat{p}_i = \frac{1}{2N} \sum_{m=1}^2 \sum_{n=1}^N x_{mn,i}$, and $E(\hat{p}_i) = p_i$, $i = 1 \dots A$. Assuming that alleles are sampled independently from the population

$$\begin{aligned} E(x_{mn,i}^2) &= p_i \\ E(x_{mn,i}x_{m'n',i}) &= E(x_{mn,i}x_{m'n',i}) = p_i^2 + \sigma_{x_{mn,i}x_{m'n',i}} \\ &= p_i^2 + p_i(1 - p_i)\theta \end{aligned}$$

where $\sigma_{x_{mn,i}x_{m'n',i}}$ is the intraclass covariance for the indicator variables and

$$\theta = \frac{\sigma_{p_i}^2}{p_i(1 - p_i)} \quad (12)$$

is the scaled among population variance in allele frequency in the populations from which this population was sampled. Using (12) we find after some algebra

$$\sigma_{\hat{p}_i}^2 = p_i(1 - p_i)\theta + \frac{p_i(1 - p_i)(1 - \theta)}{2N} .$$

²⁹This is even worse than the last time. I include it for completeness only. I really don't expect anyone (unless they happen to be a statistician) to be able to understand these details. I wouldn't recommend spending time trying to understand this unless you really, really want to understand the mathematical underpinnings of Weir and Cockerham's statistics. I've explained the fundamental principles in the text. This is just a lot of algebra, which admittedly entertains some of us who have a perverse fascination with these things.

Method	F_{is}	F_{st}	F_{it}
Direct	0.1372	0.2143	0.3221
Nei	0.3092	0.2395	0.4746
Weir & Cockerham	0.5398	0.0387	0.5577

Table 2: Comparison of Wright’s F -statistics when ignoring sampling effects with Nei’s G_{ST} and Weir and Cockerham’s θ .

Notation	
Wright	Weir & Cockerham
F_{it}	F
F_{is}	f
F_{st}	θ

Table 3: Equivalent notations often encountered in descriptions of population genetic structure.

The hat on $\sigma_{\hat{p}_i}^2$ indicates the *sample* variance of allele frequencies among populations. A natural estimate for θ emerges using the method of moments when an analysis of variance is applied to indicator variables derived from samples representing more than one population.

Applying G_{st} and θ

If we return to the data that motivated this discussion, the results in Table 2 show what we get from analyses of the $GOT-1$ data from *Isotoma petraea* (Table 1). But first a note on how you’ll see statistics like this reported in the literature. It can get a little confusing, because of the different symbols that are used. Sometimes you’ll see F_{is} , F_{st} , and F_{it} . Sometimes you’ll see f , θ , and F . And it will seem as if they’re referring to similar things. That’s because they are. They’re really just different symbols for the same thing (see Table 3). Strictly speaking the symbols in Table 3 are the *parameters*, i.e., values in the population that we try to estimate. We should put hats over any values estimated from data to indicate that they are estimates of the parameters, not the parameters themselves. But we’re usually a bit sloppy, and everyone knows that we’re presenting estimates, so we usually leave off the hats.

An example from Wright

Hierarchical analysis of variation in the frequency of the Standard chromosome arrangement of *Drosophila pseudoobscura* in the western United States (data from [1], analysis from [13]). Wright uses his rather peculiar method of accounting for sampling error. I haven't gone back to the original data and used a more modern method of analysis.³⁰

66 populations (demes) studied. Demes are grouped into eight regions. The regions are grouped into four primary subdivisions.

Results

Correlation of gametes within individuals relative to regions (F_{IR}):	0.0444
Correlation of gametes within regions relative to subdivisions (F_{RS}):	0.0373
Correlation of gametes within subdivisions relative to total (F_{ST}):	0.1478
Correlation of gametes in sample (F_{IT}):	0.2160

$$1 - F_{IT} = (1 - F_{IR})(1 - F_{RS})(1 - F_{ST})$$

Interpretation

There is relatively little inbreeding within regions ($F_{IR} = 0.04$) and relatively little genetic differentiation among regions within subdivisions ($F_{RS} = 0.04$). There is, however, substantial genetic differentiation among the subdivisions ($F_{ST} = 0.15$).

Thus, an explanation for the chromosomal diversity that predicted great local differentiation and little or no differentiation at a large scale would be inconsistent with these observations.

Reich's f -statistics

No, that heading isn't a typo. I've described Wright's F -statistics to you, but there are some other f -statistics you may encounter and should know about.³¹ The f -statistics I'll describe briefly now were introduced by Reich and colleagues [6] in the context of estimating

³⁰Sounds like it might be a good project, doesn't it? We'll see.

³¹Remember when I told you that population geneticists commonly use the same symbol or letter to refer to different things. This is another example of that. At least in this case one uses an uppercase F and the other uses a lowercase f .

the population history and admixture of human populations on the Indian subcontinent.³² If you read the paper, you won't find the definitions in the main text. They're in Supplement 1.

Reich's f -statistics are defined only for markers that have two alleles, like SNP loci. I'm going to use notation that's a bit different from that in [6], but it will match more closely what we've been using here, and it should be easier to follow. Let n_k be the number of 0 alleles in the sample from population k and let m_k be the number of 1 alleles in the sample from population k . $p_k = n_k/(n_k + m_k)$ is both an unbiased and a maximum-likelihood estimate of p_k . Let's define

$$f_4(1, 2, 3, 4) = (p_1 - p_2)(p_3 - p_4) \quad .$$

If you stare at that a bit,³³ you may recognize that f_4 looks like a correlation coefficient. In fact, as Reich et al. point out, if the true population phylogeny looks like the one in Figure 1, then the expected value of f_4 is 0. Looking at the figure you can see that populations 1 and 2 have a common history that is independent of populations 3 and 4 (and vice versa). As a result, if you calculate f_4 statistics from a large number of loci, you can see whether the relationship among four populations is consistent with the phylogeny in Figure 1.

We can also define

$$f_3(1, 2, 3) = (p_1 - p_2)(p_1 - p_3) - \frac{p_1(1 - p_1)}{n_1} \quad ,$$

where n_1 is the sample size in population 1. The extra term, $\frac{p_1(1-p_1)}{n_1}$ makes f_3 an unbiased estimator. f_3 measures how much common history p_2 and p_3 share that is independent of p_1 . If the true population phylogeny looks like the one in Figure 2, then

$$\begin{aligned} E(f_3(1, 2, 3)) &> E(f_3(2, 1, 3)) \\ E(f_3(1, 2, 3)) &> E(f_3(3, 1, 2)) \quad . \end{aligned}$$

Finally we can define

$$f_2(1, 2) = (p_1 - p_2)^2 - \frac{p_1(1 - p_1)}{n_1} - \frac{p_2(1 - p_2)}{n_2} \quad ,$$

³²The software that makes f -statistics useful in admixture analysis, `AdmixTools` (<https://github.com/DReichLab/AdmixTools>), requires that you have a C compiler, that you know how to use make to compile executables from C source code, and that you know how to install the libraries on which `AdmixTools` depends. Because of all of those requirements, we won't be using `AdmixTools` in this course. If you have data where it might be useful, I encourage you to explore it and to explore the R package `admixr`, which provides a convenient interface for it.

³³And if you remember how a correlation coefficient is defined.

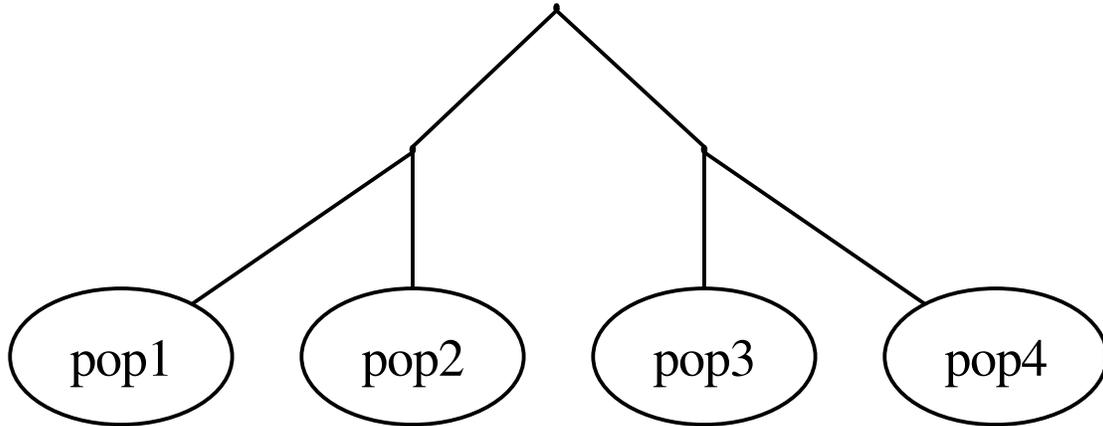


Figure 1: Hypothetical phylogeny for four populations. If you're wondering why it's upside down, I warned you. Population geneticists look at the world backward from other people. It's conventional to draw phylogenies this way in population genetics, probably because, being geneticists, population geneticists tend to think of them as pedigrees.

which gives an unbiased estimate of squared allele frequency differences between populations, analogous to pairwise F -statistics.

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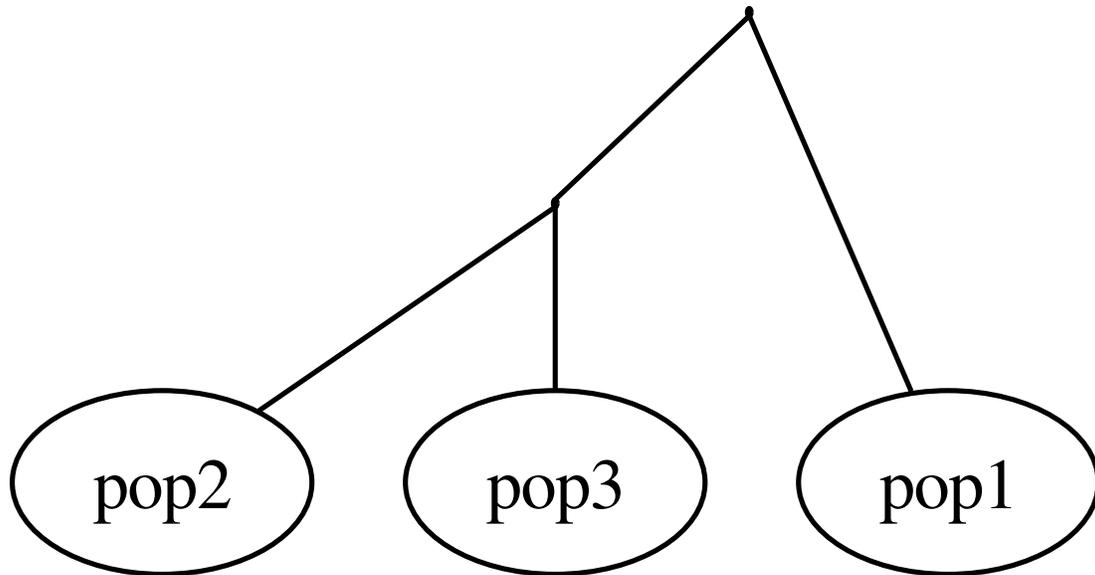


Figure 2: Hypothetical phylogeny for three populations.

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